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SOME STATISTICAL ASPECTS OF FATIGUE FAILURE THROUGH CRACK PROPAGATION

by Earl J. Brown*
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Abstract

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The objective of this work is to develop the elements of a statistical approach to fatigue failure through an analysis of growth of fatigue cracks to final failure. It is assumed that the loadings experienced by the cracked body and its material properties are of a random nature. Using concepts of fracture mechanics and a fracture mechanics approach to fatigue crack propagation, expressions are developed for the probability that a cracked body will survive a given period of loading without catastrophic failure. Author

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Statement of the Problem

Consider a cracked body loaded in some symmetric fashion about the crack as shown in Figure 1.

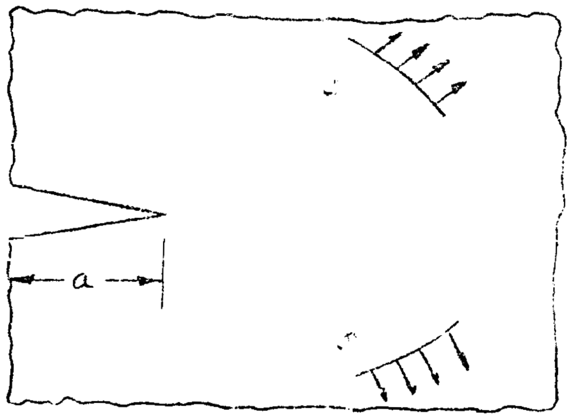

FIGURE 1.

The loadings shall be assumed to be proportional to some parameter, $s = s(t)$, which varies in a random fashion with time about a mean value $\bar{s}$ as shown in Figure 2.

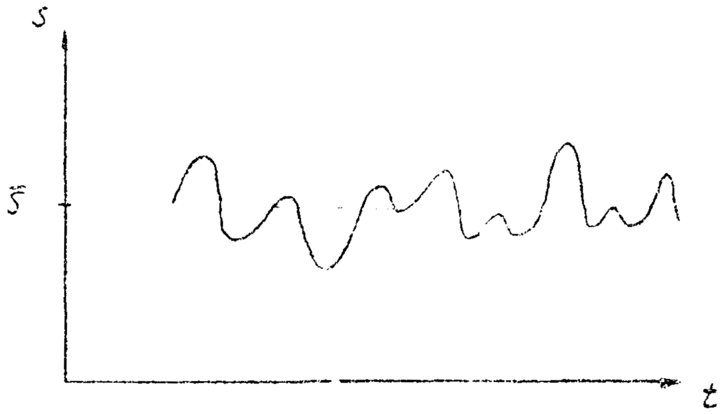

FIGURE 2.

The crack tip stress intensity factor, k, for this configuration and loading is obtained from a solution to the relevant elastic boundary value problem [1,2]. Coming from linear elastic theory, k, can always be expressed in the in the form [1,3]

$$k = k(t) = s(t)\ f(a) \tag{1}$$

where f(a) is some function of the crack length, a, as obtained from the elasticity solution.

Now, as the load, s(t), varies with time, the crack will grow at some rate depending on the statistical variation of the stress intensity factor k(t), [4]. As the crack progresses, the factor f(a) will in general increase. Consequently, the magnitude of the variation of the stress intensity factor experienced at the crack tip will increase until finally k(t) at some point in time exceeds the critical stress intensity factor, k_{cr}, causing catastrophic failure. The critical intensity factor, k_{cr}, may be regarded as a statistically described material constant (in the same sense as a yield stress or ultimate stress).

The proposed problem is, therefore, of an inherently statistical nature in computing the probability of the cracked body surviving a given period of loading, the expected lifetime, and variance in lifetime.

Survival Probabilities

In the development to follow, it will be necessary to have an expression for the probability, p(n), where

p(n) = Probability that the stress intensity factor, k_n, experienced in the $n\underline{th}$ load peak of $s(t)$ will exceed the critical stress intensity factor.

Thus $p(n)$ is seen to be the probability that the $n\underline{th}$ load peak will be sufficiently great to cause failure. Suppose that probability densities for k_n and k_{cr} are respectively $g_n(\alpha)$ and $h(\beta)$ where

$$g_n(\alpha)d\alpha = \text{Probability } \alpha < k_n < \alpha + d\alpha \tag{2}$$

$$h(\beta)d\beta = \text{Probability } \beta < k_{cr} < \beta + d\beta \tag{3}$$

Then, since the stress intensity factor in the $n\underline{th}$ load peak and the critical stress intensity factor are clearly independent, the product $g_n(\alpha)h(\beta)d\alpha d\beta$ is the joint probability that $\alpha < k_n < \alpha + d\alpha$ and $\beta < k_{cr} < \beta + d\beta$. By integrating the joint probability density over all area of the $\alpha\beta$ plane for which $\alpha > \beta$ (that is, over all points for which $k_n > k_{cr}$), one obtains the probability $p(n)$ that the stress intensity factor in the $n\underline{th}$ load peak will be sufficiently great to cause failure. The region of integration is shown as the shaded area A_s in Figure 3.

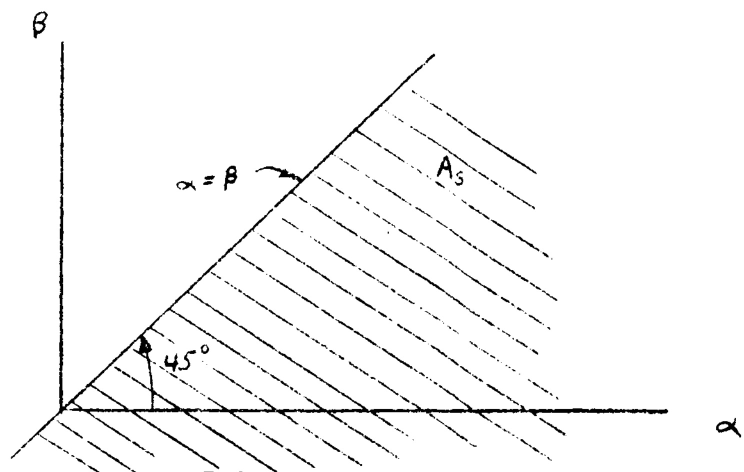

FIGURE 3

Thus,

$$p(n) = \iint_{A_s} g_n(\alpha)h(\beta)\,d\alpha\,d\beta \tag{4}$$

By carrying out the integration in different ways, either of the following two equivalent expressions can be used to compute p(n):

$$p(n) = \int_{-\infty}^{+\infty} g_n(\alpha \left\{\int_{-\infty}^{\alpha} h(\beta)d\beta\right\} d\alpha = \int_{0}^{+\infty} g_n(\alpha)\text{Prob}\left\{k_{cr}<\alpha\right\} d\alpha \tag{5}$$

$$p(n) = \int_{-\infty}^{+\infty} h(\beta)\left\{\int_{\beta}^{\infty} g_n(\alpha)d\alpha\right\} d\beta = \int_{0}^{+\infty} h(\beta)\text{Prob}\left\{k_n>\beta\right\} d\beta, \tag{6}$$

where it is noted that for physical reasons (k_{cr} cannot be negative) $h(\beta) = 0$ for $\beta < 0$, which permits changing the lower limit of $-\infty$ to a lower limit of 0.

Having an expression for p(n), one can now proceed to compute the probability P(n) that failure will actually occur in the nth load peak, and the probability F(n) that the cracked body will survive the first n load peaks. Obviously,

$$F(n) = 1 - \sum_{j=1}^{n} P(j) \tag{7}$$

Let p(n| no prior occurence) be the probability that the nth load peak will be sufficiently great such that k_n exceeds the critical stress intensity factor, given that no prior load peak stress intensity factor exceeds the critical value. Then, by the law of conditional probability,

$$P(n) = p(n \mid \text{no prior occurence})\; F(n-1) \tag{8}$$

such that

$p(x)dx$ = Probability that the loading experienced in the interval $x<n<x+dx$ will be sufficiently great to cause failure.

The quantity $F(n)$ now has the meaning

$F(n)$ = Probability that the cracked body will survive all loading between 0 and n.

The quantity $P(n)$ becomes a probability density such that

$P(x)dx$ = Probability that failure will actually occur for $x<n<x+dx$.

Further, one has the relations

$$F(n) = 1 - \int_0^n P(r)dr \tag{13}$$

or

$$\frac{dF}{dn} = -P(n) \tag{14}$$

The conditional probability statement equivalent to equation (8) is now

$$P(n)dn = p(n)dn\, F(n) \tag{15}$$

By using equation (14) and noting again that $F(0)=1$, one has

$$\frac{dF}{dn} + p(n)\, F = 0, \text{with } F(0) = 1 \tag{16}$$

Thus, the solution for $F(n)$, the probability of surviving the

As long as the magnitude of the load peak required to cause failure is sufficiently high, the heights of load peaks of interest may be considered uncorellated due to their statistical rarity. Consequently, one may assume

$$p(n|\text{ no prior occurence}) = p(n), \tag{9}$$

where p(n) is as defined earlier.

Noting that $P(n) = \bar{P}(n-1)-\bar{P}(n)$, equation (3) can be rewritten, in view of (9), as

$$\bar{P}(n) = \left\{1-p(n)\right\}\bar{P}(n-1) \tag{10}$$

This provides a recurrence relation for $\bar{P}(n)$ in terms of $\bar{P}(n-1)$. By noting that $\bar{P}(0) = 1$, that is, that the probability of surviving no loads is unity, one obtains for a solution

$$\bar{P}(n) = \prod_{j=1}^{n}\left\{1-p(j)\right\}, \tag{11}$$

and from (3) and (9),

$$P(n) = p(n)\prod_{j=1}^{n-1}\left\{1-p(j)\right\} \tag{12}$$

The preceding expressions are not particularly useful when one considers very large values of n, due to the computational difficulties involved. This difficulty can be avoided by considering n as a continuous rather than a discrete variable. Then p(n) is computed in the same way as in equations (4), (5) and (6), but is now a probability density first n load peaks, is simply the solution of (16):

$$F(n) = \exp\left\{-\int_0^n p(x)dx\right\}, \tag{17}$$

and from (15) the probability density for the occurence of failure is

$$P(n) = p(n)\exp\left\{-\int_0^n p(x)dx\right\} \tag{18}$$

An expression for the expected number of load peaks at failure, $\bar{n}$, is obtained from (14) and an integration by parts

$$\bar{n} = \int_0^\infty n\,P(n)dn = \int_0^\infty F(n)dn \tag{19}$$

For the variance from this expected value one has

$$E\left\{(n-\bar{n})^2\right\} = \int_0^\infty (n-\bar{n})^2 P(n)dn = 2\int_0^n n\,F(n)dn-\bar{n}^2 \tag{20}$$

Expressions in Terms of Crack Length

The expressions developed by p(n), the probability density for a load sufficiently great to cause failure, can be most conveniently expressed in terms of the crack length, a, instead of load peak number n. When expressed in terms of crack length, let $p(n) = p(a)$. Then

$$p(n)dn = p(a)\,\frac{da}{G}, \tag{21}$$

where $G = G(a) = da/dn$ is the crack growth rate or, more precisely, the ensemble average crack extension per load peak.

Present experimental evidence on crack growth rates indicates that G depends only on the variation of the stress intensity factor k(t) experienced at the crack tip [5]. Thus, for a given random load parameter, s(t), one can experimentally determine an expression for G in the form

$$G = G(a) = \sum_{j=1}^{N} g_j \left\{ f(a) \right\}^j \tag{22}$$

where the constants g_j will depend only on material properties and statistical constants of the random load parameter s(t). The usefulness of such an expression is that for two cracked bodies with loadings distributed differently (that is, with different expressions for f(a)), but having the same random load parameter s(t), equation (22) will hold for both cracked bodies, once experimentally determined for one of them. It should also be mentioned that the information presently available on the plasticity near a crack tip indicates that plastic effects vary with the square of the stress intensity factor; thus it is likely that only even powers should occur in the polynomial of equation (22).

Turning now to the computation of p(a), an expression will be developed for $\text{Prob}\{k_n > \beta\}$ which appears in equation (6). Consider k(t) as a random time function, and define

$$r(\delta,\gamma)d\delta d\gamma = \text{Probability that } \delta < k'(t) < \delta + d\delta \text{ and}$$

$$\gamma < k''(t) < \gamma + d\gamma, \tag{23}$$

and

$$r(\alpha,\delta,\gamma)d\alpha d\delta d\gamma = \text{Probability that } \alpha < k(t) < \alpha + d\alpha,$$

$$\delta < k'(t) < \delta + d\delta, \text{ and } \gamma < k''(t) < \gamma + d\gamma. \tag{24}$$

Then the probability that k_n, the stress intensity factor in the nth load peak, is greater than β is defined by the conditional probability statement

$$\int_{-\infty}^{0}\int_{0}^{|\gamma|d\tau} r(\delta,\gamma)\,d\delta\,d\gamma \;\; \text{Prob}\{k_n>\beta\} = \int_{\beta}^{\infty}\int_{-\infty}^{0}\int_{0}^{|\gamma|d\tau} r(\alpha,\delta,\gamma)\,d\delta\,d\gamma\,d\alpha \tag{25}$$

The triple integral on the right is the probability of having a maximum in k(t) in some time interval $d\tau$ with $k(t)>\beta$, and the double integral on the left is the joint probability of having a maximum in k(t) in some time interval $d\tau$. Prob $k_n>\beta$ is clearly the conditional probability that, given a maximum in k(t), that $k(t)>\beta$. Upon rearrangement and an integration on δ,

$$\text{Prob}\{k_n>\beta\} = \frac{\int_{\beta}^{\infty}\int_{0}^{-\infty} \gamma\, r(\alpha,0,\gamma)\,d\gamma\,d\alpha}{\int_{0}^{-\infty} \gamma\, r(0,\gamma)\,d\gamma} \tag{26}$$

The integrals in equation (26) are given in Reference [6] for the case of a stationary Gaussian process. The integral in the denominator is simply the expected number of maxima p per unit time:

$$\int_{0}^{-\infty} \gamma\, r(0,\gamma)\,d\gamma = \frac{1}{2\pi}\left(\frac{d_{33}}{d_{22}}\right)^{\frac{1}{2}} \tag{27}$$

where, d_{22} and d_{33} are the second and fourth moments respectively, of the power spectral density for k(t). The integral in the numerator cannot be given a closed form expression for all values of β, but an expression can be given for large β. This will suffice since it can safely be assumed that in the period of growth of a fatigue crack the occurence of a value

k_n greater than k_{cr} will be a statistical rarity, and thus correspond to large values of β. The result for large β is

$$\int_{\beta}^{\infty}\int_{0}^{-\infty} \gamma\, r(\alpha, 0, \gamma)\, d\gamma\, d\alpha = \frac{1}{2\pi}\left(\frac{d_{22}}{d_{11}}\right)^{1/2} \exp\left\{-\frac{(\beta-\bar{k})^2}{2d_{11}}\right\}, \qquad (28)$$

where d_{11} is the area under the power spectral density for $k(t)$ (that is, the variance of $k(t)$), and $\bar{k}$ is the mean value of $k(t)$. Thus the expression for $\text{Prob}\{k_n > \beta\}$ becomes

$$\text{Prob}\{k_n > \beta\} = \left(\frac{d_{22}^2}{d_{11}\, d_{33}}\right)^{1/2} \exp\left\{-\frac{(\beta-\bar{k})^2}{2\, d_{11}}\right\} \qquad (29)$$

It is now a simple matter to put this expression into terms involving the statistics of the load parameter, $s(t)$, and the crack length, a. Since $k(t) = s(t)\, f(a)$, one has

$$\bar{k} = \bar{s}\, f(a) \qquad (30)$$

$$d_{11} = \{f(a)\}^2\, d_{11}^{(s)} \qquad (31)$$

$$d_{22} = \{f(a)\}^2\, d_{22}^{(s)} \qquad (32)$$

$$d_{33} = \{f(a)\}^2\, d_{33}^{(s)} \qquad (33)$$

where $d_{11}^{(s)}$, $d_{22}^{(s)}$, and $d_{33}^{(s)}$ are the zeroth, second, and fourth moments of the power spectral density for $s(t)$. Let $\mu(\omega)$ be the power spectral density for $s(t)$, such that if σ_s^2 is the variance of $s(t)$, one has

$$d_{11}^{(s)} = \sigma_s^2 = \int_0^{\infty} \mu(\omega)\, d\omega \qquad (34)$$

Further, let $\overline{\omega_s^2}$ and $\overline{\omega_s^4}$ be defined by

$$d_{22}^{(s)} = \overline{\omega_s^2}\, \sigma_s^2 = \int_0^{\infty} \omega^2 \mu(\omega)\, d\omega \qquad (35)$$

and

$$d_{33}^{(s)} = \overline{\omega_s^4}\ \sigma_s^2 = \int_0^\infty \omega^4 \mu(\omega)\, d\omega \tag{36}$$

Then in view of the preceding seven equations, one has:

$$\mathrm{Prob}\{K_n > \beta\} = \frac{\overline{\omega_s^2}}{(\overline{\omega_s^4})^{1/2}} \exp\left\{- \frac{[\beta - \bar{s}\, f(a)]^2}{2\sigma_s^2 [f(a)]^2}\right\} \tag{37}$$

Thus, where $h(\beta)$ is the probability density for k_{cr},

$$p(a) = \frac{\overline{\omega_s^2}}{(\overline{\omega_s^4})^{1/2}} \int_0^\infty h(\beta) \exp\left\{- \frac{[\beta - \bar{s}\, f(a)]^2}{2\sigma_s^2 [f(a)]^2}\right\} d\beta \tag{38}$$

The preceding equation accomplishes the goal of expressing the probability density for an excess of k_{cr} in terms of crack length, a, and the statistics of the load parameter $s(t)$. Turning now to survival probabilities, define $F(n) = F(a|a_0)$. Recall that $F(n)$ is the probability of surviving n load peaks or, in terms of crack length and with the notation $F(a|a_0)$, the probability that the crack will reach a length a without unstable propagation, given that it had a length a_0 when the loading began. By use of equations (17) and (20), one obtains for the survival probability

$$F(a|a_0) = \exp\left\{- \int_{a_0}^{a} p(a) \frac{da}{G(a)}\right\} \tag{39}$$

For the probability density of failure $P(n) = P(a|a_0)$, one has, since $P(a|a_0) = -F'(a|a_0)$, (from equation (14)),

$$P(a|a_0) = \frac{p(a)}{G(a)} \exp\left\{- \int_{a_0}^{a} p(a) \frac{da}{G(a)}\right\} \tag{40}$$

The expected crack length at failure is

$$\bar{a} = \int_{a_0}^{\infty} p(a|a_0)\, a\, da \tag{41}$$

and a similar formula follows for the variance of a at failure.

The preceding developments have shown that a complete description of failure statistics under random loadings is available if one knows the two functions G(a) and p(a). The former must at the present time be experimentally determined, whereas, the latter can be predicted simply from a knowledge of the function f(a), the statistics of the load parameter s(t), and the density h(β) for k_{cr}.

Expressions in Terms of Experimentally Determined Material Parameters

In view of the preceding theoretical description, some important applications may be cited. In the aircraft industry, inspection for presence of cracks is sometimes used to estimate the residual life of structures subjected to atmospheric turbulence. This introduces two related problems: (1) If inherent cracks exist (a finite size but unseen) can the number of load peaks required for the crack to grow to an observable length be predicted? (2) After detection, can the expected crack length at failure or can a probability of surviving a certain time be predicted? Hopefully, the answers can be obtained from equations (38), (39), (40) and (41), which provide a means of analysis for these problems.

Proceeding, it is observed that the probability of survival $F(a|a_o)$, equation (39), and the expected crack length at failure $\bar{a}$, equation (41), are functions of the probability density, $h(\beta)$, for k_{cr}, equation (38). It is considered that experimental results for $h(\beta)$ can reasonably be expressed in one of the three alternative ways shown below (or others if necessary).

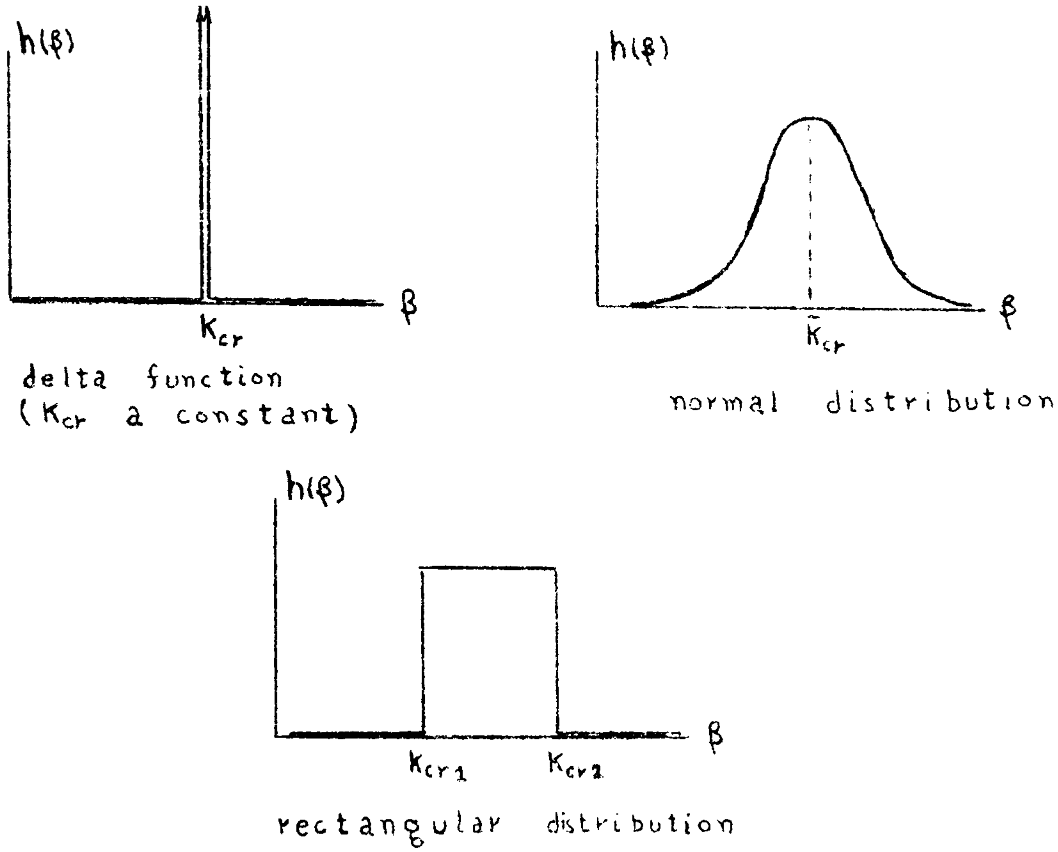

FIGURE 4.

The probability density functions in Figure 4 can be expressed as:

$$h(\beta) = \delta(\beta - K_{cr}) \qquad \text{(delta function)} \tag{42}$$

$$h(\beta) = \frac{1}{\sqrt{2\pi}\,\sigma_\beta} \exp\left\{-\frac{(\beta - \bar{K}_{cr})^2}{2\,\sigma_\beta^2}\right\} \qquad \text{(normal distribution)} \tag{43}$$

and

$$h(\beta) = \begin{cases} \dfrac{1}{K_{cr2} - K_{cr1}} & \text{for} \quad K_{cr1} < \beta < K_{cr2} \\ 0 & \text{for} \quad \beta < K_{cr1} \text{ and } \beta > K_{cr2} \end{cases} \tag{44}$$

(rectangular distribution)

In the preceding equations, $\overline{k}_{cr}$ is the mean value of the critical stress intensity factor and σ_β^2 is the variance of the normal distribution of the stress intensity factor.

It is now appropriate to consider each probability density function $h(\beta)$ individually to determine, $p(a)$, the corresponding probability density for a load sufficiently great to cause failure. In addition, for the case of $h(\beta) = \delta(\beta - k_{cr})$ the probability of survival $F(a|a_o)$ will also be determined assuming $G(a)$, equation 21, has a fourth power dependence. Moreover, with $p(a)$ known, one can at least numerically intergrate equation (39) to determine $F(a\backslash a_o)$ for the remaining cases.

Case I: Consider $h(\beta) = \delta(\beta - k_{cr})$. Then from equation (38), $p(a) =$

$$\frac{\overline{\omega_s^2}}{(\overline{\omega_s^4})^{1/2}} \int_0^\infty \delta(\beta - k_{cr}) \exp\left\{-\frac{[\beta - \bar{s}\, f(a)]^2}{2\sigma_s^2 [f(a)]^2}\right\} d\beta$$

Integrating,

$$p(a) = \frac{\overline{\omega_s^2}}{(\overline{\omega_s^4})^{1/2}} \exp\left\{-\frac{[K_{cr} - \bar{s}\, f(a)]^2}{2\sigma_s^2 [f(a)]^2}\right\} \tag{45}$$

Case II: The normal distribution for $h(\beta)$. Combining equations (38) and (43) we obtain

$$p(a) = \frac{\overline{\omega_s^2}}{(\overline{\omega_s^4})^{1/2}} \frac{1}{\sqrt{2\pi}\,\sigma_\beta} \int_0^\infty \exp\left\{-\left[\frac{[\beta - \bar{s} f(a)]^2}{2\sigma_s^2[f(a)]^2} + \frac{(\beta - \bar{k}_{cr})^2}{2\sigma_\beta^2}\right]\right\} d\beta \qquad (46)$$

If one assumes $h(\beta)$ is small except near the mean value, $\bar{k}_{cr}$, the limits of equation (46) may be extended over $-\infty < \beta < +\infty$. Combining the exponential terms and changing variable to $v = \beta - c_1$, gives,

$$p(a) = \frac{\overline{\omega_s^2}}{(\overline{\omega_s^4})^{\frac{1}{2}}} \frac{1}{\sigma_\beta\sqrt{2\pi}} \exp\left\{-\frac{[\bar{k}_{cr} - \bar{s} f(a)]^2}{2(\sigma_\beta^2 + \sigma_s^2[f(a)]^2)} \frac{1}{\sigma_\beta^2\sigma_s^2[f(a)]^2}\right\} \int_{v=-\infty}^{\infty} \exp[-c_1 v^2]\, dv,$$

where $c_1 = \dfrac{\sigma_\beta^2 + \sigma_s^2[f(a)]^2}{2\sigma_\beta^2\sigma_s^2[f(a)]^2}$.

Evaluating the integral and rearranging terms, one obtains

$$p(a) = \frac{\overline{\omega_s^2}}{(\overline{\omega_s^4})^{\frac{1}{2}}} \left[\frac{1}{1 + \frac{\sigma_\beta^2}{\sigma_s^2[f(a)]^2}}\right]^{\frac{1}{2}} \exp\left\{-\frac{[\bar{k}_{cr} - \bar{s} f(a)]^2}{2(\sigma_\beta^2 + \sigma_s^2[f(a)]^2)}\right\} \qquad (47)$$

One would expect p(a) for $h(\beta)$ a normal distribution to reduce to p(a) for $h(\beta)$ a delta function as a limiting case. In equation (47), it is observed that with $\sigma_\beta = 0$, and $\bar{k}_{cr} \to k_{cr}$, the normal distribution reduces to the sharp k_{cr} (delta function) expression for p(a).

Case III: Combining equations (38) and (44), one obtains

$$p(a) = \left[\frac{1}{k_{cr_2} - k_{cr_1}}\right] \frac{\overline{\omega_s^2}}{(\overline{\omega_s^4})^{\frac{1}{2}}} \int_{k_{cr_1}}^{k_{cr_2}} \exp\left\{-\frac{[\beta - \bar{s} f(a)]^2}{2\sigma_s^2[f(a)]^2}\right\} d\beta$$

Squaring the term in the exponential and defining

$$F^2 = \sigma_S^2 [f(a)]^2 \quad , \quad g = \frac{\bar{S}}{\sigma_S^2 f(a)} \quad and \quad \frac{v}{F\sqrt{2}} = \frac{\beta - F^2 g}{F\sqrt{2}}$$

the above equation becomes

$$p(a) = \frac{\overline{\omega_S^2}}{(\overline{\omega_S^4})^{\frac{1}{2}}} \frac{F\sqrt{2}}{K_{cr_2} - K_{cr_1}} \left\{ \int_0^{\frac{K_{cr_2} - F^2 g}{F\sqrt{2}}} \exp\left\{-\frac{v^2}{2F^2}\right\} d\left(\frac{v}{F\sqrt{2}}\right) - \int_0^{\frac{K_{cr_1} - F^2 g}{F\sqrt{2}}} \exp\left\{-\frac{v^2}{F\sqrt{2}}\right\} d\left(\frac{v}{F\sqrt{2}}\right) \right\} \quad (48)$$

Equation (46) is the standard form for the error function

$erf\ x = \frac{2}{\sqrt{\pi}} \int_0^x \exp\{-t^2\} dt$. Hence,

$$p(a) = \frac{\overline{\omega_S^2}}{(\overline{\omega_S^4})^{\frac{1}{2}}} \frac{F}{K_{cr_2} - K_{cr_1}} \sqrt{\frac{\pi}{2}} \left[erf\left(\frac{K_{cr_2} - F^2 g}{F\sqrt{2}}\right) - erf\left(\frac{K_{cr_1} - F^2 g}{F\sqrt{2}}\right) \right] \quad (49)$$

In the limit as $(K_{cr_2} - K_{cr_1}) \to 0$, the above expression for p(a), equation (48) or (49), should approach equation (45). To show this, consider equation (48) as

$$p(a) = \frac{\overline{\omega_S^2}}{(\overline{\omega_S^4})^{\frac{1}{2}}} \frac{F\sqrt{2}}{K_{cr_2} - K_{cr_1}} \int_{\frac{K_{cr_1} - F^2 g}{F\sqrt{2}}}^{\frac{K_{cr_1} + \Delta K - F^2 g}{F\sqrt{2}}} \exp\left(-\frac{v^2}{2F^2}\right) d\left(\frac{v}{F\sqrt{2}}\right)$$

$$= \frac{\overline{\omega_S^2}}{(\overline{\omega_S^4})^{\frac{1}{2}}} \frac{F\sqrt{2}}{\Delta k} \exp\left\{-\left[\frac{K_{cr_1} - F^2 g}{F\sqrt{2}}\right]^2\right\} \frac{\Delta K}{F\sqrt{2}}$$

where $\Delta k = k_{cr_2} - k_{cr_1}$

Then, in the limit, as $k_{cr_1} \to k_{cr}$

$$p(a) = \frac{\overline{\omega_S^2}}{(\overline{\omega_S^4})^{\frac{1}{2}}} \exp\left\{-\frac{[K_{cr} - \bar{S} f(a)]^2}{2\ \sigma_S^2\ [f(a)]^2}\right\} \quad (50)$$

which is identical to equation (45).

The probability density p(a) has now been determined for three different probability densities for $h(\beta)$ of k_{cr}. Recall equation (39), that the probability of survival was expressed as a function of p(a) and the crack growth rate, G(a).

Therefore, if an expression for $G(a)$ can be determined, it is possible to find the probability of survival $P(a/a_o)$. Consider a central crack in an infinite sheet for which experimental investigations [4] have shown $G(a) = \dfrac{[f(a)]^4 \overline{h_s^4}}{M} = \dfrac{a^2 \overline{h_s^4}}{M}$, where $\overline{h_s^4}$ is the average of the fourth power of the rises and falls of the load time history $s(t)$, $f(a) = \sqrt{a}$ and M is a material constant. Substituting this expression for $G(a)$ and $p(a)$ for Case I, equation (45), into equation (39) yields

$$F(a/a_o) = \exp\left\{-\int_{a_o}^{a} \frac{\overline{\omega_s^2}}{(\overline{\omega_s^4})^{1/2}} \frac{M}{\overline{h_s^4}} \frac{1}{u^2} \exp\left[-\frac{(k_{cr} - \bar{s}\sqrt{u})^2}{2\sigma_s^2 u^2}\right] du\right\} \tag{51}$$

where u represents a variable of integration corresponding to crack length, a.

Expanding the exponential term in an infinite series, the integral may be expressed as

$$\int_{u=a_o}^{a} \frac{1}{u^2} \exp\left[-\frac{(c+b\sqrt{u})^2}{g u}\right] du = \sum_{n=0}^{\infty} \int_{u=a_o}^{a} (-1)^n \frac{(c+b\sqrt{u})^n}{n!\, g^n u^{n+2}} du$$

$$= 2\sum_{n=0}^{\infty} \int_{x=\sqrt{a_o}}^{\sqrt{a}} (-1)^n \frac{(c+bx)^n}{n!\, g^n x^{2n+3}} dx \tag{52}$$

where $k_{cr} = c$, $b = -\bar{s}$, $g = 2\sigma_s^2$ and $u = x^2$.

Then from [7], equation (52) gives

$$\int_{u=a_o}^{a} \frac{1}{u^2} \exp\left[-\frac{(c+b\sqrt{u})^2}{g u}\right] du = \sum_{n=0}^{\infty} \frac{2(-1)^{n+1}}{n!\, c^{n+2} g^n} \left[\sum_{\ell=0}^{n+1} \frac{(n+1)!\,(c+bx)^{2n+2-\ell}\,(-b)^\ell}{(n+1-\ell)!\,\ell!\,(2n+2-\ell)\,x^{2n+2-\ell}}\right]\Bigg|_{x=\sqrt{a_o}}^{x=\sqrt{a}} \tag{53}$$

Combining equations (51) and (53) the probability of survival becomes

$$F(a/a_o) = \exp\left\{ -\left[\frac{\overline{\omega_s^2}}{(\overline{\omega_s^4})^{1/2}} \frac{M}{h_s^4} \right] \sum_{n=0}^{\infty} \frac{2\,(-1)^{n+1}}{n!\, K_{cr}^{n+2} (2\sigma_s^2)^n} \left[\sum_{\ell=0}^{n+1} \frac{(n+1)\,(K_{cr} - \bar{s}x)^{2n+2-\ell}\, (\bar{s})^{\ell}}{(n+1-\ell)!\; \ell!\; (2n+2-\ell)\; x^{2n+2-\ell}} \right] \Bigg|_{x=\sqrt{a_o}}^{\sqrt{a}} \right\} \tag{54}$$

Note that the probability density of failure $P(a/a_o)$, equation (40), can be evaluated with the aid of equation (54). Hence,

$$P(a/a_o) = \frac{p(a)}{G(a)} F(a/a_o) \tag{55}$$

To summarize, an expression for the probability of survival $F(a/a_o)$ has been determined for a sharply defined k_{cr} for a given material. Should the stress intensity factor, k, have a normal distribution or a plateau over a certain range, the corresponding probability density function p(a) has been determined in a manner such as to allow numerical integration to obtain the following:

(1) Probability of survival $F(a/a_o)$.

(2) Probability density of failure $P(a/a_o)$.

(3) Expected crack length $\bar{a}$ at failure.

SUMMARY AND RECOMMENDATIONS FOR FURTHER WORK

It seems advisable to discuss the necessary assumptions to describe failure statistics in two categories: (a) Derivation of the general expressions for failure statistics, and (b) comptational techniques for the general equations.

(a) Equations (39), (40) and (41) furnish the general failure properties such as:

1. The probability that the crack will reach a length a without unstable propagation, given that it had a length a_o when loading began; i.e., $P(a|a_o)$, equation (39).
2. The expected crack length at failure for the same conditions as in 1; i.e., $\bar{a}$, equation (41).

The description of failure statistics depends on the functions G(a) and p(a). The term G(a) is an experimentally determined crack growth rate law that depends only on material properties and the function f(a) obtained from the elasticity solution for a given configuration. The term p(a) can be determined if one knows the function f(a), the statistics of the load parameter s(t) and the probability density, $h(\beta)$, for k_{cr}. It should be noted that the equations to describe the failure properties were derived using concepts for fracture mechanics and fundamental probability theory. The results are true in general. The difficulty encountered in calculation is caused by an inability to predict a general crack growth rate law and a probability density, $h(k)$, for k_{cr}.

(b) To investigate computational techniques, a fourth power crack growth rate law [illegible], for a centrally located crack of length 2a and uniform stress at infinity perpendicular to the crack, was assumed. It was then assumed that the probability density, $h(k)$, for k_{cr} could reasonably be expressed as (1) a sharp peak (delta function), (2) normally distributed, or (3) constant over a short range (zero outside that range). With the assumed crack growth rate law and considering each $h(k)$ individually, it was found that expressions for survival probability could be obtained in a series form or an integral form for numerical integration. The results may be classified as follows.

(I) Consider $h(k)$ for k_{cr} to be a sharp peak (delta function). The probability that a crack will reach a length, a, without unstable propagation, given that it had a length, a_o, when loading began, $P(a|a_o)$, can be expressed in series form given by equation (54). Although not calculated, the expected crack length at failure, $\bar{a}_f$, may be determined by substituting equations (40) and (45) into (41) and performing the necessary numerical integration.

(II) Consider $h(k)$ for k_{cr} to be either a normal distribution or a constant over a short range (zero outside that range).

The corresponding survival probability, $P(a|a_o)$, may be obtained by substituting equations (47(

may be obtained by substituting equations (47) and (49) respectively into equation (39) and performing the necessary numerical integration. Similarly, the corresponding expected crack length at failure, $\bar{a}$, may be determined by respectively substituting equations (40) and (47), if $h(\beta)$ is a normal distribution, or (40) and (49), if $h(\beta)$ is a constant over a short range, into equation (41) and performing the necessary numerical integration.

This investigation has revealed areas where further experimental research and verification are necessary. Reliable growth rate data for random loads would be helpful. In addition, the relative importance of the variance σ_β^2 of $h(\beta)$ should be studied; i.e., the presence of σ_β^2 in the survival property equations may have little effect on the magnitudes of final results. Finally, computational techniques may be investigated to determine the number of required terms to ensure sufficient convergence for reliable estimates of survival probability.

Any experimental program to verify results presented here should use specimen of sufficient size to ensure that the plate is predominately elastic except for small plastic zones near the crack tip.

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